The Competence Description in Micro 3 says:

Game Theory has become a central analytic tool in much economic theory, e.g. within industrial organization, macroeconomics, international economics, labor economics, public economics and political economics.

The course aims at giving the student knowledge of game theory, non-cooperative as well as cooperative, and its applications in economic models.

The student who successfully completed the course will learn the basic game theory and will be enabled to work further with advanced game theory. The student will also learn how economic problems, involving strategic situations, can be modeled using game theory, as well as how these models are solved. The course intention is thus, that the student through this becomes able to work with modern economic theory, for instance within the areas of within industrial organization, macroeconomics, international economics, labor economics, public economics and political economics.

In the process of the course the student will learn about

- *Static games with complete information*
- *Static games with incomplete information*
- *Dynamic games with complete information*
- *Dynamic games with incomplete information*
- *Basic cooperative game theory.*

For each of these classes of games, the student should know and understand the theory, and learn how to model and analyze some important economic issues within the respective game framework.

More specifically, the students should know the theory and be able to work with both normal and extensive form games. They should know, understand and be able to apply the concepts of dominant strategies, iterative elimination of dominant strategies, as well as mixed strategies. The students should know the central equilibrium concepts in non-cooperative game theory, such as Nash Equilibrium and further refinements: Subgame-Perfect Nash Equilibrium, Bayesian Nash Equilibrium, Perfect Bayesian Equilibrium. They should understand why these concepts are central and when they are used, and be able to apply the relevant equilibrium and solution concepts.

Furthermore, the students should acquire knowledge about a number of special games and the particular issues associated with them, such as repeated games (including infinitely repeated games), auctions and signaling games.

The students should also understand and be able to apply the solution concepts of cooperative game theory, such as the core and the Shapley value. Furthermore, the students should also learn the basics of bargaining theory.

To obtain a top mark in the course the student must be able excel in all of the areas listed above.

MICRO 3 EXAM January 2010
QUESTIONS WITH SHORT ANSWERS
(Here only the short answers are given, a good exercise should argue for these answers).

1. (a) Find *all* Nash equilibria in the following game

	L	R
U	$0, -1$	$-1, 2$
D	$4, 2$	$-2, 0$

Solution: The pure-strategy equilibria are (U, R), (D, L) and the mixed eq can be determined as follows:

		q	1-q
		L	R
r	U	$0, -1$	$-1, 2$
1-r	D	$4, 2$	$-2, 0$

Row player is indifferent between playing U and D if the column player is mixing with the weight q that satisfies

$$\begin{aligned} 0 * q - 1 * (1-q) &= 4q - 2(1-q) \Leftrightarrow \\ q &= 1/5. \end{aligned}$$

Row player's best response is

$$BR_1(q) = r^*(q) \begin{cases} = 0 \text{ if } q > 1/5 \text{ (strategy D)} \\ \in [0,1] \text{ if } q = 1/5 \text{ (any combination of } U \text{ and } D) \\ = 1 \text{ if } q < 1/5 \text{ (strategy U)} \end{cases}$$

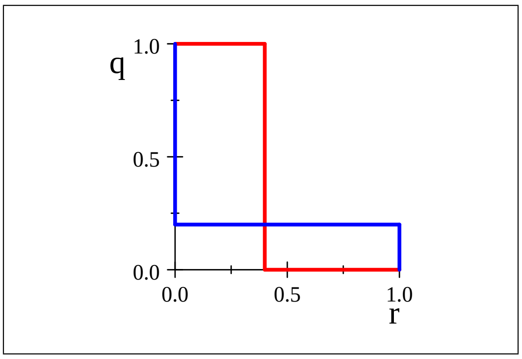
Column player is indifferent between playing L and R if the row player is mixing with the weight r that satisfies

$$\begin{aligned} -1 * r + 2(1-r) &= 2r \Leftrightarrow \\ r &= 2/5. \end{aligned}$$

Column player's best response is

$$BR_2(r) = q^*(r) \begin{cases} = 0 \text{ if } r > 2/5 \text{ (strategy R)} \\ \in [0,1] \text{ if } r = 2/5 \text{ (any combination of } L \text{ and } R) \\ = 1 \text{ if } r < 2/5 \text{ (strategy L)} \end{cases}$$

The intersection of BRs is (the BR of Player 1 is in blue, and the BR of player 2 is in red)



Therefore, the mixed strategy equilibrium is $((2/5, 3/5), (1/5, 4/5))$, i.e. the row player plays U with prob 2/5, and the column player plays L with prob 1/5.

(b) Consider the following normal-form game:

	t_1	t_2	t_3
s_1	$0, x$	$0, 3$	$1, 2$
s_2	$1, 2x$	$2, 4$	$-1, 0$

i. Under what range of values of parameter x is strategy t_1 strictly dominated?
Solution: Notice that t_3 is strictly dominated by t_2. Therefore, it is sufficient to make sure that t_2 strictly dominates t_1. This is equivalent to the following system of equations

$$\begin{aligned} x &< 3, \\ 2x &< 4, \end{aligned}$$

which yields $x < 2$. Therefore under $x < 2$ t_1 is strictly dominated (by t_2).

ii. Assume that x is such that t_1 is strictly dominated. Find all Nash equilibria of this game.
Solution: As was mentioned above, t_3 is strictly dominated by t_2. As t_1 is also strictly dominated, Player 2 will play t_2 in NE. Iterated elimination of strictly dominated strategies suggests that Player 1 will then play s_2 as it is strictly dominated by s_1. Therefore, there is a unique NE of this game, (s_2, t_2).

(c) Consider the extensive-form game represented by the game tree on Figure 1:

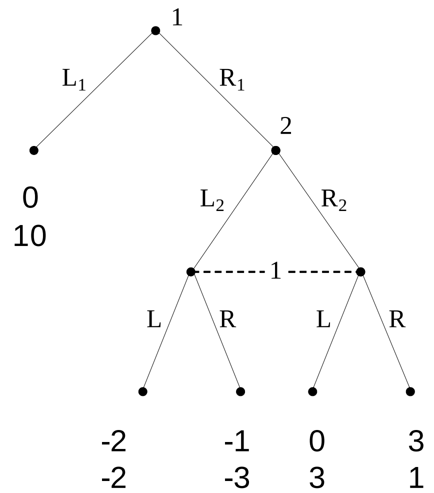


Figure 1

i. How many subgames are there in this game? Find all subgame perfect Nash equilibria.
Solution: There are two subgames including the game itself. There is unique SPNE (R_1R, R_2) (which can be found, for example, by rewriting in the normal form the subgame that starts in the node controlled by player 2).

ii. Find all pure-strategy Nash equilibria (Hint: rewrite the game in a normal form). Comment *based on the game in question* where does the difference between your answers to (i) and (ii) (if any) come from.
Solution: Normal form of the above game is given by

$$\begin{array}{ccc} & L_2 & R_2 \\ L_1L & 0,10 & 0,10 \\ L_1R & 0,10 & 0,10 \\ R_1L & -2,-2 & 0,3 \\ R_1R & -1,-3 & 3,1 \end{array} . \quad (1)$$

There are three pure-strategy NE: (L_1L, L_2),(L_1R, L_2) and (R_1R, R_2), and only the last of them is SPNE. The first two are associated with an non-credible threat exerted by Player 2: she claims she would play L_2, which is not individually rational. However, in these equilibria the threat is off equilibrium path, so it is consisted with NE. No non-credible threats, however, can be consistent with SPNE, as in SPNE individuals behave rationally in any subgame both on and off game path.

(d) Can a strictly dominated strategy be part of SPNE? If yes, suggest an example. If no, explain why not. (Be short and precise).

Solution: A strictly dominated strategy cannot be part of NE, and therefore cannot be part of SPNE, as all SPNE are NE.

2. Two firms, firm 1 and firm 2, are competing on the market in Cournot-fashion. The inverse demand function in this market is given by

$$P = a - (q_1 + q_2),$$

where q_i, $i = 1, 2$, are the output levels set by the firms, $a > 0$, and P is the price. Both firms have constant marginal costs $c > 0$ (where $a > c$). The profit of firm i is thus

$$\pi_i(q_1, q_2) = (a - (q_1 + q_2))q_i - cq_i, \quad i = 1, 2.$$

The owner of firm 1 manages her firm herself. She chooses q_1 to maximize the profit of her firm $\pi_1(q_1, q_2)$. The owner of firm 2 hires a manager instead of managing the firm herself. She provides the manager with an incentive package based on firm 2's profits and firm 2's sales q_2. More precisely, the payment to the manager, which equals his payoff is given by

$$u_m(q_1, q_2, \alpha) = 0.1 * [\pi_2(q_1, q_2) + \alpha q_2],$$

where α is the relative importance of firm 2's sales level in the manager's incentive package. The owner of firm 2 chooses α in order to maximize her income, i.e. firm 2's profit minus the payment to the manager

$$\begin{aligned} & \pi_2(q_1, q_2) - u_m(q_1, q_2, \alpha) \\ = \; & \pi_2(q_1, q_2) - 0.1 * [\pi_2(q_1, q_2) + \alpha q_2] \end{aligned}$$

The timing of the game is as follows: First, the owner of firm 2 chooses α (i.e., the composition of the incentive package). Then, both the owner of firm 1 and the manager of firm 2 observe α and simultaneously choose q_1 and q_2, respectively.

(a) What are the levels of output chosen by the owner of firm 1 and the manager of firm 2 in the second stage of the game? How do they depend on α? Provide an intuition for your answer.

Solution: In the second stage both the owner of firm 1 and the manager take α as given. The owner of firm 1 solves

$$\max_{q_1} (a - (q_1 + q_2))q_1 - cq_1.$$

FOC is

$$a - 2q_1 - q_2 - c = 0,$$

and the best response of the owner of firm 1 is

$$q_1 = \frac{a - c - q_2}{2}. \tag{2}$$

The manager of firm 2 solves

$$\begin{aligned}\max_{q_2} u_m(q_1, q_2, \alpha) &= \max_{q_2} 0.1 * [\pi_2(q_1, q_2) + \alpha q_2] \\ &= \max_{q_2} 0.1 * [(a - (q_1 + q_2))q_2 - cq_2 + \alpha q_2] .\end{aligned}$$

FOC is

$$0.1 * [a - 2q_2 - q_1 - c + \alpha] = 0,$$

and the best response of the manager is

$$q_2 = \frac{a - c + \alpha - q_1}{2} \tag{3}$$

Solving (2) and (3) together yields the levels of output chosen by the owner of firm 1 and the manager of firm 2 in the second stage

$$\begin{aligned}q_1(\alpha) &= \frac{a - c - \alpha}{3} \\ q_2(\alpha) &= \frac{a - c + 2\alpha}{3}\end{aligned}$$

The higher is α, the more output is produced by the second firm and the less output is produced by the first firm. The reason is that higher α implies that the manager of firm 2 is compensated more for higher sales (i.e. q_2). So she becomes more "aggressive" on the market, which has to be accommodated by the owner of firm 1.

(b) Find the level of α chosen by the owner of firm 2 in the subgame perfect equilibrium and find the levels of output in the subgame perfect equilibrium? Comment on how the strategic market position of firm 2 is affected by hiring a manager with an incentive package.

Solution: The owner of firm 2 solves the following maximization problem in the first stage:

$$\begin{aligned}&\max_{\alpha} \pi_2(q_1(\alpha), q_2(\alpha)) - 0.1 * [\pi_2(q_1(\alpha), q_2(\alpha)) + \alpha q_2(\alpha)] \\ =\ &\max_{\alpha} 0.9\pi_2(q_1(\alpha), q_2(\alpha)) - 0.1\alpha q_2(\alpha) \\ =\ &\max_{\alpha} 0.9 [(a - (q_1 + q_2))q_2 - cq_2] - 0.1\alpha q_2 \\ &s.to \quad q_1(\alpha) = \frac{a - c - \alpha}{3} \\ &\qquad\quad q_2(\alpha) = \frac{a - c + 2\alpha}{3}\end{aligned}$$

This can be further rewritten as

$$\begin{aligned}&\max_{\alpha} \left[0.9 \left(a - \frac{2(a - c) + \alpha}{3} - c\right) - 0.1\alpha\right] \frac{a - c + 2\alpha}{3} \\ \Leftrightarrow\ &\max_{\alpha} \left[0.9\frac{(a - c)}{3} - 0.4\alpha\right] \frac{a - c + 2\alpha}{3}\end{aligned}$$

FOC yields

$$\begin{aligned}-0.4 * \frac{a - c + 2\alpha}{3} + \left[0.9\frac{(a - c)}{3} - 0.4\alpha\right] \frac{2}{3} = 0 \Leftrightarrow \\ \alpha = \frac{(a - c)}{8}.\end{aligned}$$

Substituting it into the levels of output found in (a), we get

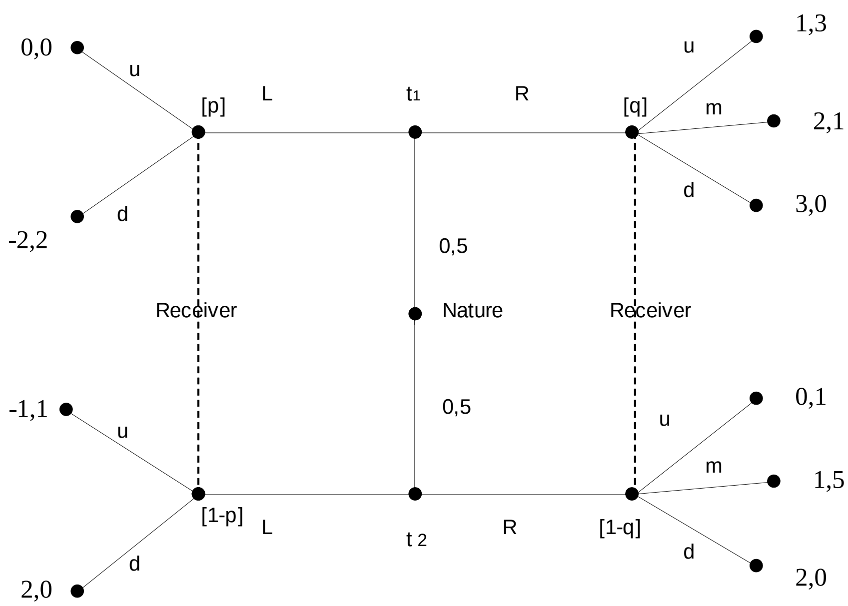
$$q_1 = \frac{a-c-\alpha}{3} = \frac{7}{24}(a-c),$$
$$q_2 = \frac{a-c+2\alpha}{3} = \frac{5}{12}(a-c).$$

We know that in the simple Cournot equilibrium with linear demand each firm produces 1/3 of total market capacity. Here firm 2 ends up with

$$q_2 = \frac{5}{12}(a-c) > \frac{4}{12}(a-c) = \frac{(a-c)}{3}.$$

Therefore, providing the manager with an incentive package based on sales results in firm 2 occupying larger share of the market, as compared to Cournot benchmark. Notice however, that it is not necessarily profitable for the owner of firm 2 to hire a manager - on the one hand, firm 2 is now credibly producing more. It is likely to provide firm 2 with higher profits. On the other hand, the owner of firm 2 has to pay the manager, so the net difference in firm 2 owner's payoffs is not necessarily positive (and requires further calculations).

3. Consider the following signalling game.



(a) Find a pooling perfect Bayesian equilibrium or show that there is none.

Solution: Notice that no matter what the Receiver chooses in the right information set, the payoff of type 1 of Sender will be higher from choosing R, then from choosing L. Therefore the only possible pooling equilibrium is when senders pool on R. Assume they do. SR 3 implies that $q = 1/2$. In the right information set the payoff of the Receiver from choosing u is then

$$3 * 1/2 + 1 * 1/2 = 2,$$

the payoff from choosing m is

$$1 * 1/2 + 5 * 1/2 = 3,$$

the payoff from choosing d is

$$0 * 1/2 + 0 * 1/2 = 0.$$

SR2 then implies that the Receiver chooses m in the right information set. As we mentioned above, type 1 of the Sender will never deviate from R. If the Receiver chooses d in the left information set, it cannot be an equilibrium, as type 2 of the Sender will then deviate from R. Therefore, in order to have an equilibrium we should have the Receiver choose u in the left info set. The payoff of the Receiver from choosing u in the right set is

$$0 * p + 1 * (1 - p) = 1 - p,$$

the payoff from choosing d is

$$2 * p + 0 * (1 - p) = 2p.$$

For Receiver to choose u we should have

$$\begin{aligned} 1 - p &> 2p \Leftrightarrow \\ p &< 1/3 \end{aligned}$$

Therefore, there are many pooling PBE in this case, all of the following type: $(RR, um, (p, 1-p), (1/2, 1/2))$ such that $p < 1/3$.

(b) Find a separating perfect Bayesian equilibrium or show that there is none.

Solution: For the same reason as above (playing R dominates playing L for type 1 of Sender) the only possible separating PBE is when type 1 of the Sender plays R, and type 2 plays L. In this case by SR 3

$$\begin{aligned} p &= 0, \\ q &= 1, \end{aligned}$$

which implies that the Receiver should play u in the left info. set and d in the right info. set. However this is inconsistent with PBE, as type 2 of the Sender would then deviate to R. Therefore there is no separating PBE in this game.

4. Two brothers, Anders and Thomas, are bargaining about the way of sharing the 500-gram cake their mother has baked for them. Anders' utility from consuming x_A grams of cake is given by

$$u_A(x_A) = x_A.$$

Thomas' utility from getting x_T grams of cake is given by

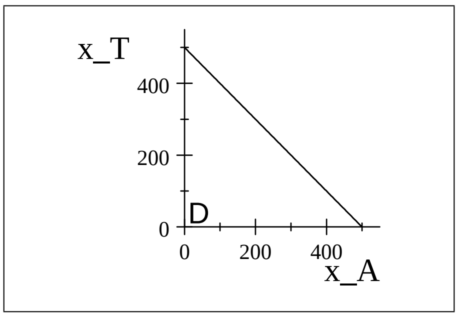
$$u_T(x_T) = 2x_T.$$

Their mother told them that if they fail to reach an agreement, she will give the cake to their neighbors, so that neither of brothers receives anything.

(a) Represent the situation as a bargaining problem, i.e. draw the sets X and U, and mark the disagreement points. Describe the efficient allocations.

Solution:

$$X = \{x_A, x_T : x_A + x_T \leq 500,\ x_i \geq 0\}, \quad D = (0, 0)$$

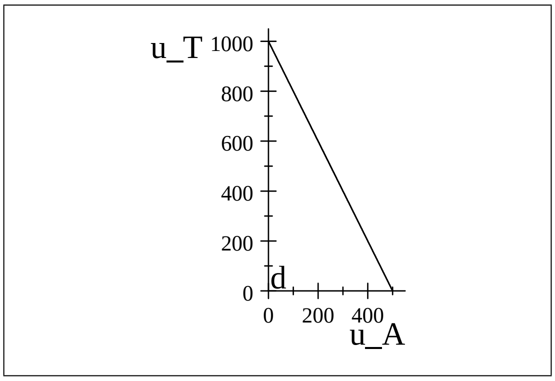


Based on the expression for set X we can write

$$U = \{u_A(x_A), u_T(x_T) : x_A + x_T \leq 500, x_i \geq 0\}.$$

As $u_A(x_A) = x_A$ and $u_T(x_T) = 2x_T$, we can rewrite it as

$$U = \{u_A(x_A), u_T(x_T) : u_A + u_T/2 \leq 500, u_i \geq 0\}$$
$$d = (u_A(D), u_T(D)) = (0, 0)$$



The efficient allocations are all allocations where

$$u_A + u_T/2 = 500, u_i \geq 0,$$

that is, where the no cake is left over, everything is divided.

(b) Determine the Nash bargaining solution of the game.
Solution: Let's solve for it analytically:

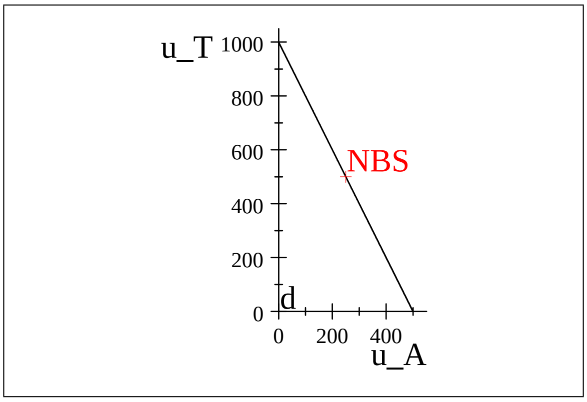
$$\max_{u_A,\, u_T} (u_A - d_A)(u_T - d_T)$$
$$s.to \quad u_i > d_i$$
$$u_A + u_T/2 \leq 500$$

or, as $d_i = 0$

$$\max_{u_A,\, u_T} u_A u_T$$
$$s.to \quad u_i > 0$$
$$u_A + u_T/2 \leq 500$$

Solving it we get

$$u_T^{NBS} = 500,$$
$$u_A^{NBS} = 250.$$



(c) Assume now that once Anders has eaten enough of the cake, he is satiated and does not enjoy more cake. More precisely, he does not get any extra utility from consuming more than 300 grams of cake, that is, his utility is now given by

$$u_A(x_A) = \begin{cases} x_A, & \text{if } x_A \leq 300 \\ 300, & \text{if } x_A > 300 \end{cases}.$$

Thomas' utility is still $u_T(x_T) = 2x_T$. Draw the modified set U, and apply the Nash bargaining solution axioms to your answer in (b) to find the Nash bargaining solution of this problem.

Solution: Now the set U will be bounded by the bold line in the plot below. Notice that the NBS of the original problem belongs to the new, smaller U-set, so by the 4th axiom ("independence of irrelevant alternatives") it will also be the NBS of the new problem.